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Analytic Model for Aircraft Survivability Assessment of a One-on-One Engagement

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Aircraft survivability assessment is one of the most important steps of the aircraft survivability project. Many models were developed to assess one-on-one engagement aircraft survivability; however, most of them belong in the simulation category. Based on the theory of stochastic duel, this paper constructs an analytic model for engagement-level aircraft survivability assessment. The analytic model explicitly represents the encounter process between the aircraft and the weapon system, such as target detection, acquisition and firing, and reloading and firing. The general solution of aircraft survival probability is obtained in quadrature form and the explicit expression is derived from a particular detection-time distribution and a firing-time distribution. According to this model, not only can the aircraft survival probability be calculated, but the survivability contour maps can also be easily plotted with the mean time of detection, the single-shot kill probability, and other parameters. The numerical examples and analysis of the results show that the model is reasonable and effective.

Nomenclature

a	=	duration of the preparation phase (preparation time)
C	=	characteristic transform
d	=	generic time interval
f	=	generic function
f_d	=	probability density function of the detection time
f_k	=	probability density function of the firing time
G	=	probability function that the aircraft is killed during a period of engagement
h	=	probability density function of the time to kill
i	=	imaginary number
P_S	=	probability that the aircraft survives the one-on-one engagement
P_{SSK}	=	single-shot kill probability
q_{SSK}	=	complementary of the single-shot kill probability
r_d	=	reciprocal of the mean time of detection
r_k	=	reciprocal of the mean time between two shots (the average rate of fire)
S	=	event that the aircraft survives the encounter
s_1	=	time that the aircraft spends before it enters the lethal envelope
s_2	=	time that the aircraft spends before it leaves the lethal envelope
t	=	time, values of various random variables
t_1	=	time that the weapon's sensor takes to detect the aircraft (detection time)
u	=	transform variable in the characteristic transform
δ	=	impulse symbol
Φ_k	=	characteristic function of the distribution of the weapon's time to kill
φ_d	=	characteristic function of the detection-time distribution
φ_k	=	characteristic function of the firing-time distribution

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I. Introduction

AIRCRAFT combat survivability is defined as the capability of an aircraft to avoid or withstand a man-made hostile environment [1]. Aircraft survivability assessment refers to the modeling and qualification of events and elements in the encounter between the aircraft and the weapon system. It is one of the most important steps of the aircraft survivability project. The engagement-level survivability assessment is the basis of assessments of other combat levels (mission and campaign). Over the past decades, many models and computer programs were developed to assess aircraft survivability in one-on-one engagements. Some of those programs were widely used and distributed, such as RADGUNS (Radar Directed Gun Simulation) [2], ESAMS (Enhanced Surface-to-Air Missile Simulation) [3], and TRAP (Trajectory Analysis Programs) [4]. However, all of these programs belong in the simulation category, and few analytic models of aircraft survivability assessment have been presented and published. Sometimes analytic models have advantages over simulation models in terms of speed of calculation and ease of use, and analytic models are easier and more convenient for examining the effects of parameters on analytical results. This paper attempts to construct an analytic model for engagement-level aircraft survivability assessment that employs the mathematical model called the stochastic duel.

Williams and Ancker [5] are the chief contributors who have cast light upon the development of the theory of stochastic duel. In a stochastic duel, two contestants fire at each other until one is killed, and the time between rounds fired is a random variable. The term *marksman problem* is used to describe one contestant firing at a passive target. The fundamental duel is formed by two independent marksmen against one another. Ancker [6] continued the development of the theory of stochastic duels to include the case in which there is a time limitation on the duration of the duel. However, the factor concerning the time required to detect the target is not considered in Williams and Ancker's [5] model. It was Wand et al. [7] who introduced the process of detection into stochastic duel. Recent studies by Smith [8] have suggested that the theory of stochastic duel can be used to analyze tank survivability.

In this study, the weapon system and the aircraft are taken to be the contestant and the target, respectively. The encounter between the contestant (weapon system) and target (aircraft) is modeled as a marksman problem, but this marksman problem model is not the same as that of Williams and Ancker [5]. It is assumed that the contestant must detect and acquire the target before it shoots at the target, and the contestant's detection and shot are limited within certain envelopes.

II. Scenario of the one-on-one Engagement

A. Big Picture for the One-on-One Encounter

Consider the aircraft and the surface-based weapon system shown in Fig. 1. The aircraft flies along a specific flight route through the territory defended by the surface-based weapon system. The detection envelope and the lethal envelope of the weapon system are indicated.

The detection envelope represents the capability of the weapon system's sensor to detect the presence of a target. The extent of the detection envelope is a function of the weapon's sensor parameters and the aircraft's signatures. The lethal envelope is the extent that a weapon can reach out and kill the target. There are two kinds of lethal envelopes: lethal launch envelope and lethal intercept envelope. The lethal launch envelope is based on the location of the aircraft at the time that the threat propagator launched (fired). The lethal intercept envelope is based upon the location of the aircraft at the time of the threat propagator intercept. The lethal launch envelope is used in this paper. It is assumed that detection and lethal envelopes are known; the method of estimating the actual extents of those envelopes for the weapon systems such as antiaircraft artillery (AAA) and surface-to-air missiles will not be discussed here.

B. Process of the Encounter

The entire process of the encounter between the aircraft and the weapon system can be divided into three subsequent phases: detection, preparation, and engagement.

1. Detection Phase

The detection phase refers to the process in which the weapon system searches for and detects the aircraft. When an aircraft flies through the detection envelope of a weapon system, perhaps the aircraft will be detected at some time, but no a priori prediction regarding the specific time the detection occurs can be made. So it is assumed that the time taken to detect, t_1, is a random variable with a known probability density function $f_d(t_1)$.

2. Preparation Phase

The preparation phase includes the process in which the weapon system acquires the presence of the aircraft, tracks the aircraft for a while, and finally obtains a fire control solution. That is to say, at the end of the preparation phase, the weapon system is armed and ready to shoot at the aircraft. In this paper, *preparation time* refers to the duration of the preparation phase, denoted by a. It is taken to be a constant performance parameter.

3. Engagement Phase

The engagement phase begins when the weapon system shoots at the aircraft for the first time. The first shot is taken only when the following conditions apply:

1) The weapon performed the detection and preparation phase.

2) The aircraft is exposed to the lethal envelope.

Assume that the weapon fires at the aircraft in a shoot–look–shoot mode; that is to say, there is a look to determine if the aircraft has survived after the first shot. A second shot is taken if the look reveals that the aircraft was not killed by the first shot. And more shots might be fired if the aircraft has survived the second shot and is still within the lethal envelope of the weapon. The engagement phase ends when the aircraft leaves the lethal envelope.

One of the key parameters of the engagement phase is the firing time (the time between shots). It consists of the time of flight of the projectile or missile, time of target kill assessment, and time of reloading and firing. The firing time is taken to be a random variable with a known distribution $f_k(t)$. Successive firing time are selected from $f_k(t)$ independently and at random. Each time the weapon system shoots, it has a fixed probability P_{SSK} of killing the aircraft. Based on the theory of stochastic duel [5,6], the probability that the aircraft is killed between the time t and $t + dt$ after the first shot is given as

$$h(t)dt = P_{\mathrm{SSK}} f_k(t)dt + P_{\mathrm{SSK}} q_{\mathrm{SSK}} f_k(t) * f_k(t)dt + P_{\mathrm{SSK}} q_{\mathrm{SSK}}^2 f_k(t) * f_k(t) * f_k(t)dt + \cdots \tag{1}$$

where $*$ denotes the convolution of the firing-time distribution functions.

The probability that the aircraft is killed after time d has elapsed from the first shot may be expressed as

$$G(d) = \int_0^d h(t)\mathrm{d}t \tag{2}$$

III. Development of the Model

There are three possible outcomes of the one-on-one encounter between the weapon system and the aircraft:

1) The weapon system is ready to shoot after the aircraft leaves the lethal envelope.

2) The weapon system is ready to shoot when the aircraft is in the lethal envelope.

3) The weapon system is ready to shoot before the aircraft enters the lethal envelope.

Consider the scenario of Fig. 1; assume that the aircraft enters the detection envelope at time 0, the time that the aircraft spends before it enters the lethal envelope is s_1, and the time the aircraft spends before it leaves the lethal envelope is s_2. The time that the weapon system spends in getting ready to shoot consists of two intervals: detection time t_1 and preparation time a. Thus, the preceding three outcomes correspond to three mathematical expressions, respectively: 1) $t_1 + a > s_2$, 2) $s_1 \leq t_1 + a \leq s_2$, and 3) $t_1 + a < s_1$.

The analytic results of the probability that the aircraft survives the one-on-one encounter are derived by separately considering the aircraft survival probabilities of the three outcomes.

A. $t_1 + a > s_2$

In this situation, the weapon system spends too much time on the detection and preparation phases, and as the weapon system prepares to fire at the aircraft, the aircraft has left the lethal envelope. It makes the survival of the aircraft a certainty, and so the conditional probability that the aircraft survives the encounter, given that $t_1 + a > s_2$, is

$$P(S|t_1 + a > s_2) = 1 \tag{3}$$

The probability that $t_1 + a > s_2$ is given by

$$P(t_1 + a > s_2) = P(t_1 > s_2 - a) = \int_{s_2-a}^{+\infty} f_d(t_1)\mathrm{d}t_1 \tag{4}$$

From Eqs. (3) and (4), the probability that $t_1 + a > s_2$ and the aircraft survives can be expressed as

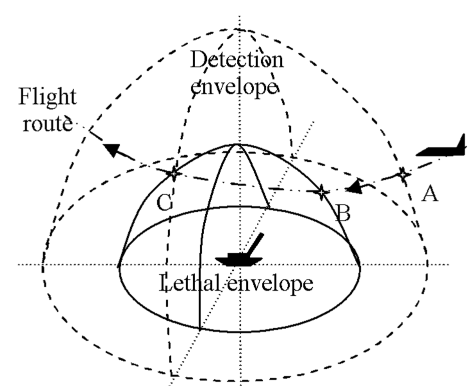

Fig. 1 One-on-one encounter.

$$P(t_1 + a > s_2 \text{ and } S) = P(t_1 + a > s_2) \cdot P(S|t_1 > s_2 - a) = \int_{s_2-a}^{+\infty} f_d(t_1)\mathrm{d}t_1 \tag{5}$$

B. $s_1 \leq t_1 + a \leq s_2$

When $s_1 \leq t_1 + a \leq s_2$, the extent of t_1 can be expressed as $[s_1 - a, s_2 - a]$. The probability that the detection time plus preparation time lies between the interval from $t_1 + a$ to $t_1 + a + dt_1$ is $f_d(t_1)dt_1$. In this situation, the weapon system shoots at the aircraft for the first time at $t_1 + a$ and does not stop the engagement until the aircraft leaves the lethal envelope. The probability that the aircraft survives the first shot is $1 - P_{\text{SSK}}$. The probability that the aircraft survives the engagement of the interval from $t_1 + a$ to s_2 is $1 - G(s_2 - t_1 - a)$. Thus, the probability that detection time plus preparation time falls between the interval from $t_1 + a$ to $t_1 + a + dt_1$ and the aircraft survives is given by

$$dP(t_1 + a \text{ and } S) = (1 - P_{\text{SSK}})[1 - G(s_2 - t_1 - a)] \cdot f_d(t_1)dt_1 \tag{6}$$

The probability that $s_1 \leq t_1 + a \leq s_2$ and the aircraft survives can be expressed as the integral of t_1 over the extent of $[s_1 - a, s_2 - a]$:

$$\begin{aligned} P(s_1 \leq t_1 + a \leq s_2 \quad \text{and} \quad S) &= \int_{S_1-a}^{S_2-a} \mathrm{d}P(t_1 + a \text{ and } S) \\ &= \int_{s_1-a}^{s_2-a} (1 - P_{\text{SSK}})[1 - G(s_2 - t_1 - a)]f_d(t_1)\mathrm{d}t_1 \end{aligned} \tag{7}$$

C. $t_1 + a < s_1$

When the weapon system is ready to shoot before the aircraft enters the lethal envelope, the weapon system will not fire until the aircraft enters the lethal envelope. This implies that the weapon system shoots at the aircraft for the first time while the aircraft enters the lethal envelope. After that, the weapon system will not stop the engagement until the aircraft leaves the lethal envelope. The conditional probability that the aircraft survives the encounter, given that $t_1 + a < s_1$, is

$$P(S|t_1 + a < s_1) = (1 - P_{\text{SSK}})[1 - G(s_2 - s_1)] \tag{8}$$

The probability that $t_1 + a < s_1$ is

$$P(t_1 + a < s_1) = P(t_1 < s_1 - a) = \int_0^{s_1-a} f_d(t_1)\mathrm{d}t_1 \tag{9}$$

From Eqs. (8) and (9), the probability that $t_1 + a < s_1$ and the aircraft survives is given by

$$P(t_1 + a < s_1 \text{ and } S) = (1 - P_{\text{SSK}})[1 - G(s_2 - s_1)] \int_0^{s_1-a} f_d(t_1)\mathrm{d}t_1 \tag{10}$$

D. Analytic Expression of the Aircraft Survivability

Taking into account the three possible outcomes of the one-on-one encounter, the probability that aircraft survives the one-on-one encounter is found from

$$\begin{aligned} P_S &= P(t_1 + a > s_2 \quad \text{and} \quad S) + P(s_1 \leq t_1 + a \leq s_2 \text{ and } S) + P(t_1 + a < s_1 \text{ and } S) \\ &= 1 - P_{\text{SSK}} \int_0^{s_1-a} f_d(t_1)\mathrm{d}t_1 - q_{\text{SSK}} G(s_2 - s_1) \int_0^{s_1-a} f_d(t_1)\mathrm{d}t_1 \\ &\quad - q_{\text{SSK}} \int_{s_1-a}^{s_2-a} f_d(t_1)G(s_2 - t_1 - a)\mathrm{d}t_1 \end{aligned} \tag{11}$$

Note that Eq. (11) holds when $a < s_1$.

When $s_1 \leq a \leq s_2$, there are only two possible outcomes for the one-on-one encounter, which are as follows:

1) The weapon system is ready to shoot after the aircraft leaves the lethal envelope.

2) The weapon system is ready to shoot when the aircraft is in the lethal envelope.

The method used to derive the analytic results has an analogy to the method described in the preceding part, and the analytic result is given by

$$\begin{aligned} P_S = 1 &- P_{\text{SSK}} \int_0^{s_2-a} f_d(t_1)\mathrm{d}t_1 \\ &- q_{\text{SSK}} \int_0^{s_2-a} f_d(t_1)G(s_2 - a - t_1)\mathrm{d}t_1 \qquad s_1 \leq a \leq s_2 \end{aligned} \tag{12}$$

When $a > s_2$, the only outcome is that the weapon system is ready to shoot after the aircraft leaves the lethal envelope. Hence,

$$P_S = 1 \qquad a > s_2 \tag{13}$$

The calculation of the integrals in Eqs. (11) and (12) is a difficult task, and the explicit form for P_S is attainable only for some simple distributions of detection time and firing time; these are, of course, serious restrictions. However, by using characteristic functions [5,6], these integrals of real functions can be transformed into integrals of complex functions, which permits the evaluation in closed form of some otherwise intractable expressions.

The characteristic function for any detection-time distribution is

$$\varphi_d(u) = \int_0^{+\infty} f_d(t_1)e^{iut_1}\mathrm{d}t_1 \tag{14}$$

The characteristic function for any firing-time distribution is

$$\varphi_k(u) = \int_0^{+\infty} f_k(t)e^{iut}\mathrm{d}t \tag{15}$$

The characteristic function for the distribution of the weapon's time to kill is

$$\Phi_k(u) = \int_0^{+\infty} h(t)e^{iut}\mathrm{d}t \tag{16}$$

From the definitions in Eqs. (15) and (16) and from the convolution property of characteristic function, Eq. (1) may be transformed into [5]

$$\begin{aligned} \Phi_k(u) &= P_{\text{SSK}}\varphi_k(u) + P_{\text{SSK}}q_{\text{SSK}}\varphi_k^2(u) + P_{\text{SSK}}q_{\text{SSK}}^2\varphi_k^3(u) + \cdots \\ &= P_{\text{SSK}}\varphi_k(u)/[1 - q_{\text{SSK}}\varphi_k(u)] \end{aligned} \tag{17}$$

Returning now to Eqs. (11) and (12), transform the integrals of real functions to the form of complex functions. From the definitions in Eq. (14) and from Parseval's theorem [9], the following integral can be expressed as

$$\int_0^{s_2-a} f_d(t_1)\mathrm{d}t_1 = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i(s_2-a)u} - 1}{u}\varphi_d(-u)\mathrm{d}u \tag{18a}$$

$$\int_0^{s_1-a} f_d(t_1)\mathrm{d}t_1 = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i(s_1-a)u} - 1}{u}\varphi_d(-u)\mathrm{d}u \tag{18b}$$

Additionally, $G(s_2 - s_1)$ can be expressed as the integral of characteristic function $\Phi_k(u)$:

$$G(s_2 - s_1) = \int_0^{s_2-s_1} h(t)\mathrm{d}t = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i(s_2-s_1)u} - 1}{u}\Phi_k(-u)\mathrm{d}u \tag{19}$$

From the properties of characteristic function, the characteristic function of $G(s_2 - a - t_1)$ can be calculated:

$$C[G(s_2 - a - t_1)] = \int_{-\infty}^{+\infty} G(s_2 - a - t_1) e^{iut_1} \mathrm{d}t_1$$
$$= e^{i(s_2-a)u} \Phi_k(-u)/iu + \pi\delta(u) \tag{20}$$

From Eqs. (14) and (20) and from the convolution property of characteristic function, the characteristic function of $f_d(t_1)G(s_2 - a - t_1)$ can be expressed as

$$C[f_d(t_1)G(s_2 - a - t_1)] = \frac{1}{2\pi} C[f_d(t_1)] * C[G(s_2 - a - t_1)]$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [e^{i(s_2-a)w}\Phi_k(-w)/iw + \pi\delta(w)]\varphi_d(u-w)\mathrm{d}w \tag{21}$$

From Eqs. (21) and from Parseval's theorem, the following integral can be expressed as

$$\int_{s_1-a}^{s_2-a} f_d(t_1)G(s_2 - a - t_1)\mathrm{d}t_1$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{e^{-iu(s_2-a)} - e^{-iu(s_1-a)}}{-iu} \int_{-\infty}^{+\infty}\left[\frac{e^{i(s_2-a)w}\Phi_k(-w)}{iw}\right.$$
$$\left. + \pi\delta(w)\right]\varphi_d(u-w)\mathrm{d}w\mathrm{d}u \tag{22a}$$

$$\int_{0}^{s_2-a} f_d(t_1)G(s_2 - a - t_1)\mathrm{d}t_1$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{e^{-iu(s_2-a)} - 1}{-iu} \int_{-\infty}^{+\infty}\left[\frac{e^{i(s_2-a)w}\Phi_k(-w)}{iw}\right.$$
$$\left. + \pi\delta(w)\right]\varphi_d(u-w)\mathrm{d}w\mathrm{d}u \tag{22b}$$

Substituting Eqs. (18a), (18b), (19), (22a), and (22b) into Eqs. (11) and (12) for P_S gives

$$P_S = 1 - \frac{P_{\mathrm{SSK}}}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i(s_2-a)u} - 1}{u}\varphi_d(-u)\mathrm{d}u$$
$$+ \frac{q_{\mathrm{SSK}}}{4\pi^2}\int_{-\infty}^{+\infty} \frac{e^{i(s_2-s_1)u} - 1}{u}\Phi_k(-u)\mathrm{d}u$$
$$\times \int_{-\infty}^{+\infty} \frac{e^{i(s_1-a)u} - 1}{u}\varphi_d(-u)\mathrm{d}u$$
$$- \frac{q_{\mathrm{SSK}}}{4\pi^2}\int_{-\infty}^{+\infty} \frac{e^{-iu(s_2-a)} - e^{-iu(s_1-a)}}{-iu}\left\{\int_{-\infty}^{+\infty}\left[\frac{e^{i(s_2-a)u}\Phi_k(-w)}{iw}\right.\right.$$
$$\left.\left. + \pi\delta(w)\right]\varphi_d(u-w)\mathrm{d}w\right\}\mathrm{d}u \qquad a < s_1 \tag{23}$$

$$P_S = 1 - \frac{P_{\mathrm{SSK}}}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i(s_2-a)u} - 1}{u}\varphi_d(-u)\mathrm{d}u$$
$$- \frac{q_{\mathrm{SSK}}}{4\pi^2}\int_{-\infty}^{+\infty} \frac{e^{-iu(s_2-a)} - 1}{-iu}\left\{\int_{-\infty}^{+\infty}\left[\frac{e^{i(s_2-a)u}\Phi_k(-w)}{iw}\right.\right.$$
$$\left. + \pi\delta(w)\right]\varphi_d(u-w)\mathrm{d}w\}\mathrm{d}u \qquad s_1 \le a \le s_2 \tag{24}$$

It seems that Eqs. (23) and (24) are more complicated than Eqs. (11) and (12), but these two equations are more convenient for calculations than Eqs. (11) and (12). The integrals of Eqs. (23) and (24) can be evaluated using residual theorem [10].

Equations (11), (12), (23), and (24) are the general solution to the one-on-one engagement and the principal results of this paper. The calculation of P_S can be divided into three steps:

1) Based on the distributions of detection time $f_d(t)$ and firing time $f_k(t)$, calculate $\varphi_d(u)$ and $\Phi_k(u)$ from Eqs. (14) and (17), separately.

2) Substitute $\varphi_d(u)$ and $\Phi_k(u)$ into Eqs. (23) and (24), evaluate the integrals, and get the explicit expressions for P_S.

3) Substitute the values of the parameters (such as P_{SSK}, a, s_1, and s_2) into the explicit expression and obtain the actual value of P_S.

IV. Examples and Discussion

A. Derivation of P_S

Let the distribution of detection time be negative exponential [11]:

$$f_d(t) = r_d e^{-r_d t} \tag{25}$$

And let the distribution of firing time be Erlang $(2, r_k)$:

$$f_k(t) = 4r_k^2 e^{-2r_k t} \tag{26}$$

The characteristic functions for Eqs. (25) and (26) are

$$\varphi_d(u) = r_d/(r_d - iu) \tag{27}$$

and

$$\varphi_k(u) = 4r_k^2/(2r_k - iu)^2 \tag{28}$$

Substituting Eq. (28) into Eq. (17) yields the characteristic function of $h(t)$:

$$\Phi_k(u) = \frac{4r_k^2 P_{\mathrm{SSK}}}{(2r_k - iu)^2 - 4r_k^2 q_{\mathrm{SSK}}} \tag{29}$$

Substituting Eqs. (27) and (29) into Eq. (23) for P_S gives

$$P_S = 1 - \frac{P_{\mathrm{SSK}}}{2\pi i}\int_{-\infty}^{+\infty} \frac{(e^{i(s_2-a)u} - 1)r_d}{u(r_d + iu)}\mathrm{d}u$$
$$+ \frac{q_{\mathrm{SSK}}}{4\pi^2}\int_{-\infty}^{+\infty} \frac{(e^{i(s_2-s_1)u} - 1)4r_k^2 P_{\mathrm{SSK}}}{u[(2r_k + iu)^2 - 4r_k^2 q_{\mathrm{SSK}}]}\mathrm{d}u$$
$$\times \int_{-\infty}^{+\infty} \frac{(e^{i(s_1-a)u} - 1)r_d}{u(r_d + iu)}\mathrm{d}u$$
$$- \frac{q_{\mathrm{SSK}}}{4\pi^2}\int_{-\infty}^{+\infty} \frac{e^{-iu(s_2-a)} - e^{-iu(s_1-a)}}{-iu}$$
$$\times \left\{\int_{-\infty}^{+\infty}\left[\frac{e^{i(s_2-a)w}4r_k^2 P_{\mathrm{SSK}}}{iw(2r_k + iw)^2 - 4r_k^2 q_{\mathrm{SSK}}}\right.\right.$$
$$\left. + \pi\delta(w)\right]\frac{r_d}{r_d - i(u-w)}\mathrm{d}w\}\mathrm{d}u \tag{30}$$

Because evaluations of the integrals in Eq. (30) are similar, a detailed process of the evaluation will be given for only one of these integrals with the remainder of the results of other integrals given directly.

In evaluating the first integral

$$\int_{-\infty}^{+\infty} \frac{(e^{i(s_2-a)u} - 1)r_d}{u(r_d + iu)}\mathrm{d}u$$

let

$$f(u) = \frac{(e^{i(s_2-a)u} - 1)r_d}{u(r_d + iu)}$$

The function $f(u)$ has a pole of order 1 at $u = ir_d$ and a removable singular point at $u = 0$. By defining

$$f(0) = \lim_{u\to 0} f(u) = i(s_2 - a)$$

a function is obtained

$$f(u) = \begin{cases} \frac{(e^{i(s_2-a)u} - 1)r_d}{u(r_d + iu)} & u \ne 0 \\ i(s_2 - a) & u = 0 \end{cases}$$

which is analytic for $-\infty < u < +\infty$.

Consider the integral taken around the closed-contour D (see Fig. 2), consisting of the line segment $-R \le u \le +R$ and the semicircle $|u| = R$. Let us take R large enough, which means that D

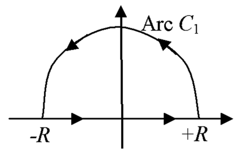

Fig. 2 Closed-contour *D*.

encloses the pole $u = ir_d$. Hence, According to the residual theorem, we can get

$$\int_{-R}^{+R} f(u)\mathrm{d}u + \int_{C_1} f(u)\mathrm{d}u = 2\pi i Res[f(u), ir_d] \tag{31}$$

Let $R \to +\infty$; the integral on the extreme left becomes the integral being evaluated, and we can show that the second integral on the left becomes zero. As R becomes large enough, we have

$$\begin{aligned}
\int_{C_1} \frac{(e^{i(s_2-a)u}-1)r_d}{u(r_d+iu)}\mathrm{d}u &\le \int_{C_1}\left|\frac{(e^{i(s_2-a)u}-1)r_d}{u(r_d+iu)}\right|\mathrm{d}s \\
&< \int_{C_1}\frac{(|e^{i(s_2-a)u}|+1)r_d}{R(R-r_d)}\mathrm{d}s = \frac{r_d}{R(R-r_d)}\int_{C_1}(e^{-(s_2-a)y}+1)\mathrm{d}s \\
&= \frac{2r_d}{(R-r_d)}\int_0^{\pi/2}(e^{-(s_2-a)R\sin\theta}+1)\mathrm{d}\theta \\
&\le \frac{2r_d}{(R-r_d)}\int_0^{\pi/2}(e^{-(s_2-a)R(2\theta/\pi)}+1)\mathrm{d}\theta \\
&= \frac{\pi r_d}{(R-r_d)}\left(\frac{1-e^{-(s_2-a)R}}{(s_2-a)R}+1\right)
\end{aligned}$$

As $R \to +\infty$, the right side of this equation goes to zero, which means that the integral on the left must also become zero. Thus,

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{(e^{i(s_2-a)u}-1)r_d}{u(r_d+iu)}\mathrm{d}u &= 2\pi i Res[f(u), ir_d] \\
&= 2\pi i \lim_{u\to ir_d}(u-ir_d)\frac{(e^{i(s_2-a)u}-1)r_d}{u(r_d+iu)} = 2\pi i(1-e^{-(s_2-a)r_d})
\end{aligned} \tag{32a}$$

By using the same method, we can evaluate other integrals in Eq. (30), yielding the following results:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\frac{(e^{i(s_2-s_1)u}-1)4r_k^2 P_{\mathrm{SSK}}}{u[(2r_k+iu)^2-4r_k^2 q_{\mathrm{SSK}}]}\mathrm{d}u \\
&= 2\pi i\{1-e^{-2r_k(s_2-s_1)}[\sinh(2r_k\sqrt{q_{\mathrm{SSK}}}(s_2-s_1)) \\
&\quad + \sqrt{q_{\mathrm{SSK}}}\cosh(2r_k\sqrt{q_{\mathrm{SSK}}}(s_2-s_1))]/\sqrt{q_{SSK}}\}
\end{aligned} \tag{32b}$$

$$\int_{-\infty}^{+\infty}\frac{(e^{i(s_1-a)u}-1)r_d}{u(r_d+iu)}\mathrm{d}u = 2\pi i(1-e^{-(s_1-a)r_d}) \tag{32c}$$

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\frac{e^{-iu(s_2-a)}-e^{-iu(s_1-a)}}{-iu}\left\{\int_{-\infty}^{+\infty}\left[\frac{e^{i(s_2-a)w}4r_k^2 P_{\mathrm{SSK}}}{iw(2r_k+iw)^2-4r_k^2 q_{\mathrm{SSK}}}\right.\right. \\
&\quad \left.\left. + \pi\delta(w)\right]\frac{r_d}{r_d-i(u-w)}\mathrm{d}w\right\}\mathrm{d}u \\
&= -\frac{2\pi^2}{\sqrt{q_{\mathrm{SSK}}}}r_d e^{-r_d(t_2-a)}\left[(1+\sqrt{q_{\mathrm{SSK}}})\frac{e^{(s_2-s_1)\beta}-1}{\beta}\right. \\
&\quad \left. -(1-\sqrt{q_{\mathrm{SSK}}})\frac{e^{(s_2-s_1)\gamma}-1}{\gamma}\right] + 4\pi^2(e^{-r_d(s_1-a)}-e^{-r_d(s_2-a)})
\end{aligned} \tag{32d}$$

where $\beta = r_d - 2r_k(1-\sqrt{q_{\mathrm{SSK}}})$, and $\gamma = r_d - 2r_k(1+\sqrt{q_{\mathrm{SSK}}})$.

Substituting Eqs. (32a–32d) into Eq. (30) for P_S gives

$$\begin{aligned}
P_S &= e^{-(s_2-a)r_d} + \frac{1}{2}\sqrt{q_{\mathrm{SSK}}}r_d e^{-r_d(s_2-a)} \\
&\times\left[(1+\sqrt{q_{\mathrm{SSK}}})\frac{e^{(s_2-s_1)\beta}-1}{\beta}-(1-\sqrt{q_{\mathrm{SSK}}})\frac{e^{(s_2-s_1)\gamma}-1}{\gamma}\right] \\
&+ \sqrt{q_{\mathrm{SSK}}}(1-e^{-(s_1-a)r_d})e^{-2r_k(s_2-s_1)}[\sinh(2r_k\sqrt{q_{\mathrm{SSK}}}(s_2-s_1)) \\
&+ \sqrt{q_{\mathrm{SSK}}}\cosh(2r_k\sqrt{q_{\mathrm{SSK}}}(s_2-s_1))] \qquad a < s_1
\end{aligned} \tag{33a}$$

The P_S for $s_1 \le a \le s_2$ can be obtained in a similar fashion:

$$\begin{aligned}
P_S &= e^{-(s_2-a)r_d} + \frac{1}{2}\sqrt{q_{\mathrm{SSK}}}r_d e^{-r_d(s_2-a)}\left[(1+\sqrt{q_{\mathrm{SSK}}})\frac{e^{(s_2-a)\beta}-1}{\beta}\right. \\
&\left. -(1-\sqrt{q_{\mathrm{SSK}}})\frac{e^{(s_2-a)\gamma}-1}{\gamma}\right] \qquad s_1 \le a \le s_2
\end{aligned} \tag{33b}$$

and

$$P_S = 1 \qquad s_2 < a \tag{33c}$$

It can be seen from Eqs. (33a–33c) that the aircraft survival probability can be expressed as a function of six parameters: r_d, P_{SSK}, s_1, s_2, a, and r_k.

B. Numerical Calculations and Discussion

1. Scenario Description and Survivability Calculation

As shown in Fig. 1, the aircraft flies a close air support mission and encounters the AAA. The radius of the detection envelope and the lethal launch envelope of the AAA are 18 and 4 km, respectively. It is assumed that the time for AAA to detect the aircraft follows a negative exponential distribution and the mean time for AAA to detect the aircraft is 10 s. The preparation time of the weapon system is 8 s. The mean time for the AAA to reload and fire is 3 s; the firing time follows Erlang (2, r_k) distribution. The speed of the aircraft is 500 m/s. The aircraft flight route is indicated in the figure. At point A, the aircraft enters the detection envelope of the AAA and then enters and leaves the lethal envelope of the AAA at points B and C, respectively. The lengths of AB and BC are 10 and 7 km. Assume that each time the weapon shoots, it has a probability of 0.2 to kill the aircraft.

From the preceding description, we know that the value of the single-shot kill probability P_{SSK} is 0.2, and the value of the preparation time a is 8 s. The reciprocal of the mean time of detection r_d is given by

$$\begin{aligned}
r_d &= 1/\text{mean time for AAA to detect the aircraft} \\
&= 1/(10\ \text{s}) = 0.1 \quad 1/\text{s}
\end{aligned}$$

The average rate of fire r_k is given by

$$\begin{aligned}
r_k &= 1/\text{mean time for the AAA to reload and fire} \\
&= 1/(3\ \text{s}) = 1/3 \quad 1/\text{s}
\end{aligned}$$

According to the lengths of flight trajectories AB and BC and the speed of the aircraft, s_1 and s_2 are given by

$$\begin{aligned}
s_1 &= \text{the length of the flight trajectory AB/the speed of the aircraft} \\
&= (10\times 10^3\ \text{m})/(500\ \text{m/s}) = 20\ \text{s}
\end{aligned}$$

$$\begin{aligned}
s_2 &= \text{the length of the flight trajectory AB} \\
&\quad + \text{BC/the speed of the aircraft} \\
&= (10+7)\times 10^3\ \text{m}/(500\ \text{m/s}) = 34\ \text{s}
\end{aligned}$$

The intermediate variables β and γ can be obtained, respectively:

$$\beta = 0.1 - 2 \cdot 1/3 \cdot (1 - \sqrt{0.8}) = 0.03$$
$$\gamma = 0.1 - 2 \cdot 1/3 \cdot (1 + \sqrt{0.8}) = -1.16$$

It is seen that $a < s_1$, and so Eq. (33a) is used to calculate the probability that aircraft survives the one-on-one engagement:

$$\begin{aligned} P_S = {} & e^{-(34-8)0.1} + \frac{1}{2}\sqrt{0.8} \cdot 0.1 \cdot e^{-0.1(34-8)} \\ & \times \left[(1+\sqrt{0.8}) \frac{e^{(34-20)0.03} - 1}{0.03} - (1-\sqrt{0.8}) \frac{e^{(34-20)\cdot(-1.16)} - 1}{-1.16} \right] \\ & + \sqrt{0.8} \cdot (1 - e^{-(20-8)0.1}) \cdot e^{-2\cdot 0.33 \cdot (34-20)} \\ & \cdot [\sinh(2 \cdot 0.33 \cdot \sqrt{0.8} \cdot (34-20)) \\ & + \sqrt{0.8}\cosh(2 \cdot 0.33 \cdot \sqrt{0.8} \cdot (34-20))] = 0.404 \end{aligned}$$

Thus, the probability that the aircraft survives the one-on-one engagement is 0.404.

2. *Influence of the Parameters to the Survival Probability*

Of interest here is the influence of the different parameters on the probability that the aircraft survives the encounter. These parameters can be divided into three groups:

1) Parameters r_d and P_{SSK} are influenced by the aircraft's basic design, ordnance, survivability equipment, and self-defense weapons that the aircraft carries to avoid the weapon system.

2) Parameters s_1 and s_2 are mainly affected by the aircraft fight route-planning tactics.

3) Parameters a and r_k depend on the performance of the weapon system.

A. Effects of the Aircraft's Performance. Substitute $s_1 = 20$, $s_2 = 34$, $a = 8$, and $r_k = 1/3$ into Eq. (33a), then P_S can be expressed as a function of r_d and P_{SSK}. Figure 3 presents the effects of varying r_d and P_{SSK}.

As one would expect, the aircraft survival probability increases as the mean time to detection increases, and it decreases along with the increase of single-shot kill probability. These plots may be used to compare the aircraft survivability results from different performances of the aircraft.

B. Effects of the Aircraft Fight Route. Substitute $r_d = 1/10$, $P_{\mathrm{SSK}} = 0.2$, $a = 8$, and $r_k = 1/3$ into Eqs. (33a–33c), then P_S can be expressed as a function of s_1 and s_2. Plots of P_S with the aircraft's duration within the detection envelope s_1 and the lethal envelope $s_2 - s_1$, are shown in Fig. 4.

Reducing the aircraft's durations within the weapon's detection envelope and the lethal envelope increases the aircraft survival probability, as shown in Fig. 4. This contour map may be helpful for the combat commander to choose aircraft flight routes.

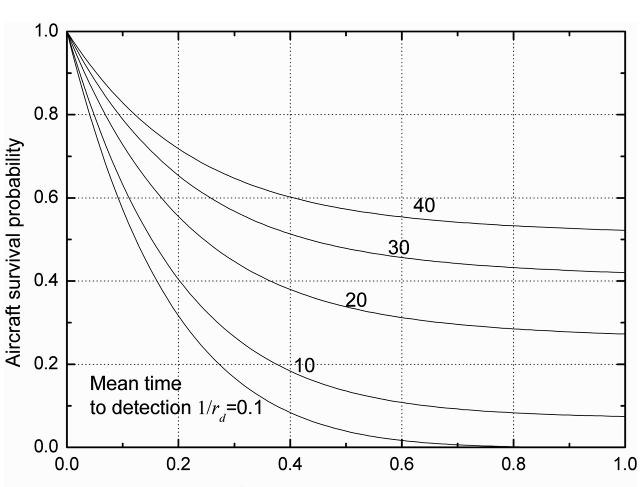

Fig. 3 Effects of the aircraft's performance.

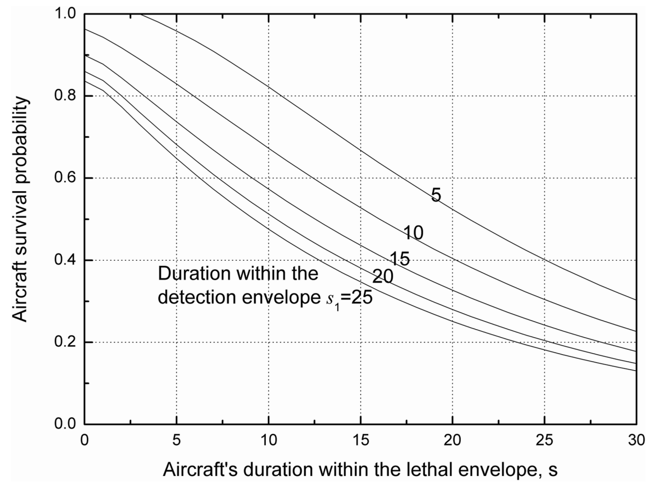

Fig. 4 Effects of the aircraft's combat tactics.

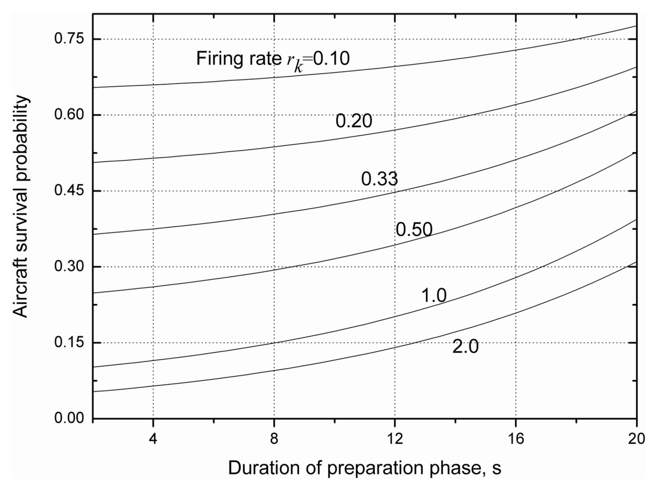

Fig. 5 Effects of the weapon's performance.

C. Effects of the Weapon's Performance. Substitute r_d= $1/10$, $P_{\mathrm{SSK}} = 0.2$, $s_1 = 20$, and $s_2 = 34$ into Eq. (33a), then P_S can be expressed as a function of a and r_k. Figure 5 shows the relationship between P_S and a and r_k.

Figure 5 shows that the aircraft survival probability increases with the increase of the duration of the weapon's preparation phase, and decreases with the increase of the firing rate of the weapon system. This figure can be used to evaluate the effectiveness of the weapon system. These results show that the analytic model for aircraft survivability assessment of a one-on-one engagement is reasonable and effective.

V. Conclusions

An analytic model for engagement-level aircraft survivability assessment has been developed. The three major features of this model are as follows:

1) This model is an analytic one, not a simulated one. The general solution of aircraft survival probability is presented in integral form, and the method to evaluate this integral using characteristic functions is also presented.

2) This model incorporates the three major factors essential to aircraft survivability. These three major factors are the threat, the aircraft, and the scenario.

3) This model represents part of the encounter process between the aircraft and the surface-based weapon system, such as target detection, acquisition and firing, and reloading and firing, but it does

not include endgame effects such as fragment or hit distribution on the aircraft and vulnerability of the aircraft.

According to the explicit expressions of aircraft survival probability, not only can the aircraft survival probability be calculated quickly, but the effects of the parameters on the aircraft survival probability can also be analyzed conveniently. This model may be useful for trade studies of aircraft survivability enhancement design, tactic selection of the aircraft combat, and analyzing the effectiveness of the surface-based weapon system.

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